Solution 6

1. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:

   (a) $[1,2) \cup (2,5) \cup \{10\}$.
   (b) $[0,1] \cap \mathbb{Q}$.
   (c) $\bigcup_{k=1}^{\infty}(1/(k+1), 1/k)$.
   (d) $\{1,2,3,\cdots\}$.

   Solution.

   (a) Boundary points: $1,2,5,10$. Interior points: $(1,2),(2,5)$. Interior: $(1,2) \cup (2,5)$. Closure: $[1,5] \cup \{10\}$.
   (b) Boundary points: All points in $[0,1]$. No interior point. Interior: the empty set ϕ. Closure: $[0,1]$
   (c) Boundary points: $\{1/k : \ k \geq 1\}, 0$. Interior points: all points in this set. Interior: This set (because it is an open set). Closure: $[0,1]$.
   (d) Boundary points $1,2,3,\cdots$. No interior points. Interior: ϕ. Closure: the set itself (it is a closed set).

2. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:

   (a) $R \equiv [0,1) \times [2,3) \cup \{0\} \times (3,5)$.
   (b) $\{(x,y) : 1 < x^2 + y^2 \leq 9\}$.
   (c) $\mathbb{R}^2 \setminus \{(1,0),(1/2,0),(1/3,0),(1/4,0),\cdots\}$.

   Solution.

   (a) Boundary points: the geometric boundary of the rectangle and the segment $\{0\} \times [3,5]$. Interior points: all points inside the rectangle. Interior $(0,1) \times (3,5)$. Closure $[0,1] \times [3,5] \cup \{0\} \times [3,5]$.
   (b) Boundary points: all (x,y) satisfying $x^2+y^2=1$ or $x^2+y^2=9$. Interior points: all points satisfying $1 < x^2+y^2 < 9$. Interior $\{(x,y) : \ 1 < x^2+y^2 < 9\}$. Closure $\{(x,y) : \ 1 \leq x^2+y^2 \leq 9\}$.
   (c) Boundary points: $(0,0),(1,0),(1/2),(1/3,0),\cdots$. Interior points: all points except boundary points. Interior: $\mathbb{R}^2 \setminus \{(0,0),(1,0),(1/2),(1/3,0),\cdots\}$. Closure: $\mathbb{R}^2$.

3. Describe the closure and interior of the following sets in $C[0,1]$:

   (a) $\{f : \ f(x) > -1, \ \forall x \in [0,1]\}$.
   (b) $\{f : \ f(0) = f(1)\}$.

   Solution.

   (a) Closure: $\{f \in C[0,1] : \ f(x) \geq -1, \ \forall x \in [0,1]\}$. Interior: The set itself. It is an open set.
   (b) Closure: The set itself. It is a closed set. Interior: ϕ. Let f satisfy $f(0) = f(1)$. For every $\varepsilon > 0$, it is clear we can find some $g \in C[0,1]$ satisfying $\|g - f\|_\infty < \varepsilon$ but $g(0) \neq g(1)$. It shows that every metric ball $B_\varepsilon(f)$ must contain some functions lying outside this set.

4. Let A and B be subsets of (X,d). Show that $\overline{A\cup B}=\overline{A}\cup\overline{B}$. Does $\overline{A\cap B}=\overline{A}\cap\overline{B}$?

   Solution. We have $\overline{A}\subset\overline{B}$ whenever $A\subset B$ right from definition. So $\overline{A}\cup\overline{B}\subset\overline{A\cup B}$. Conversely, if $x\in\overline{A\cup B}$, $B_\varepsilon(x)$ either has non-empty intersection with A or B. So there exists $\varepsilon_j\to 0$ such that $B_{\varepsilon_j}(x)$ has nonempty intersection with A or B, so $x\in\overline{A}\cup\overline{B}$.

   On the other hand, $\overline{A\cap B}=\overline{A}\cap\overline{B}$ is not always true. For instance, consider intervals (a,b) and (b,c). We have $\overline{(a,b)}\cap\overline{(b,c)}=\{b\}$ but $\overline{(a,b)\cap(b,c)}=\phi$. Or you take A to be the set of all rationals and B all irrationals. Then $\overline{A\cap B}=\overline{\phi}=\phi$ but $\overline{A}\cap\overline{B}=\mathbb{R}$!

5. Show that $\overline{E}=\{x\in X:\ d(x,E)=0\}$ for every non-empty $E\subset X$.

   Solution. Let $A=\{x\in X:\ d(x,E)=0\}$. Claim that A is closed. Let $x_n\to x$ where $x_n\in A$. Recalling that $x\mapsto d(x,E)$ is continuous, so $d(x,E)=\lim_{n\to\infty}d(x_n,E)=0$, that is, $x\in A$. We conclude that A is a closed set. As it clearly contains E, so $\overline{E}\subset A$ since the closure of E is the smallest closed set containing E. On the other hand, if $x\in A$, then $B_{1/n}(x)\cap E\neq\phi$. Picking $x_n\in B_{1/n}(x)\cap E$, we have $\{x_n\}\subset E$, $x_n\to x$, so $x\in\overline{E}$.

6. Let $E\subset(X,d)$. Show that E° is the largest open set contained in E in the sense that $G\subset E^\circ$ whenever $G\subset E$ is open.

   Solution. Let $G\subset E$ is open. For $x\in G$, there is some $B_\varepsilon(x)\subset G$. But that means $B_\varepsilon(x)\subset E$ too, so x is an interior point of E, that is, $x\in E^\circ$. We have shown $G\subset E^\circ$. Next, we claim that E° is open. For, if x is an interior point, there is some $B_r(x)\subset E$. But then every point $y\in B_r(x)$ is also an interior point as $B_\rho(y)\subset B_r(x)\subset E$ where $\rho=r-d(x,y)$.

7. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

   Solution. $\mathbb{Z}$ is a closed subset so it is complete. On the other hand, the closure of $\mathbb{Q}$ is $\mathbb{R}$, it is not complete.

8. Does the collection of all differentiable functions on $[a,b]$ form a complete set in $C[a,b]$?

   Solution. No. Since $C[a,b]$ is complete, it suffices to show that the set of differentiable functions is not closed. But this is easy, I leave you to verify the sequence of differentiable functions $f_n(x)=(1/n+x^2)^{1/2}$ in $C[-1,1]$ converges uniformly to the non-differentiable function $f(x)=|x|$.

9. Optional. Let (X,d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that it forms a complete metric space under the sup-norm. This problem will be used in the next problem.

   Solution. Let $\{f_n\}$ be a Cauchy sequence in $C_b(X)$. For $\varepsilon>0$, there exists n_1 such that

$$|f_n(x)-f_m(x)|\leq\|f_n-f_m\|_\infty<\varepsilon,\quad\forall x\in X.\tag{1}$$

   It shows that $\{f_n(x)\}$ is a numerical Cauchy sequence, so $\lim_{n\to\infty}f_n(x)$ exists. We define $f(x)=\lim_{n\to\infty}f_n(x)$. We check it is continuous at x_0 as follows. By passing $m\to\infty$ in (1), we have

$$|f(x)-f(x_0)|\leq|f(x)-f_{n_1}(x)|+|f_{n_1}(x)-f_{n_1}(x_0)|+|f_{n_1}(x_0)-f(x_0)|\leq 2\varepsilon+|f_{n_1}(x)-f_{n_1}(x_0)|.$$

   As f_{n_1} is continuous, there is some δ such that $|f_{n_1}(x)-f_{n_1}(x_0)|<\varepsilon$ for $x\in B_\delta(x_0)$. It follows that we $|f(x)-f(x_0)|<3\varepsilon$ for $x\in B_\delta(x_0)$, so f is continuous at x_0. Now, letting $m\to\infty$ in (1), we get $|f_n(x)-f(x)|\leq\varepsilon$ for all $n\geq n_1$, so $f_n\to f$ uniformly. In particular, it means f is bounded.

10. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting

$$d(n,m) = \left| \frac{1}{n} - \frac{1}{m} \right| .$$

(a) Show that it is not a complete metric.

(b) Describe how to make it complete by adding one new point.

Solution. The sequence $\{n\}$ is a Cauchy sequence in this metric but it has no limit. Its completion is obtained by adding an ideal point called ∞ and define $\tilde{d}(x,y) = d(x,y)$ when $x, y \in \mathbb{Z}$ and $\tilde{d}(x,\infty) = 0$ for all $x \in \mathbb{Z}$ or ∞.

11. Let (X, d) be a metric space. Fixing a point $p \in X$, for each x define a function

$$f_x(z) = d(z,x) - d(z,p).$$

(a) Show that each f_x is a bounded, uniformly continuous function in X.

(b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X, d) to $C_b(X)$ (shorthand for $C_b(X, \mathbb{R})$) . In other words,

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

(c) Deduce from (b) the completion theorem.

This approach is much shorter than the proof given in notes. However, it is not so inspiring.

Solution.

(a) From $|f_x(z)| = |d(z,x) - d(z,p)| \le d(x,p)$, and from $|f_x(z) - f_x(z')| \le |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)| \le 2d(z,z')$, it follows that each f_x is a bounded, uniformly continuous function in X.

(b) $|f_x(z) - f_y(z)| = |d(z,x) - d(z,y)| \le d(x,y)$, and equality holds taking $z = x$. Hence

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

(c) Let $Y_0 = \{f_x : x \in X\} \subset C_b(X)$. Let Y be the closure of Y_0 in the complete metric space $(C_b(X), \rho)$ with sup-norm ρ. Then (Y, ρ) is a completion of (X, d).

12. Optional. Let $\mathcal{K}$ be the collection of all non-empty closed and bounded sets in $\mathbb{R}^n$. We introduce a metric called the Hausdorff metric on $\mathcal{K}$ as follows. The set E_ε is defined to be the set $\{x + \varepsilon z : \ x \in E, \ |z| < 1\}$, $\varepsilon > 0$. For closed and bounded E, F, define

$$\rho_H(E,F) = \inf \{\varepsilon : \ F \subset E_\varepsilon, \ E \subset F_\varepsilon\} .$$

(a) Show that

$$E_\varepsilon = \{y \in \mathbb{R}^n : \ d(y,E) < \varepsilon \ \} .$$

(b) Show that

$$\rho_H(E,F) = \max\left\{ \sup_{x\in E} d(x,F), \ \sup_{y \in F} d(y,E) \right\} ,$$

where $d(x,F)$ is the Euclidean distance from x to F.

(c) Show that ρ_H is a metric on $\mathcal{K}$.

(d) Let $\{K_n\}, K_{n+1} \subset K_n$, be a descending sequence in $\mathcal{K}$. Show that

$$\rho_H(K_n, K_\infty) \to 0, \qquad \text{as } n \to \infty \ ,$$

where $K_\infty = \cap_j K_j \neq \phi$.

Solution. Google.

13. Optional. In the previous problem, it is shown that the Hausdorff metric makes $\mathcal{K}$, the set of all non-empty closed and bounded sets in $\mathbb{R}^n$, a metric space. Now show that it is complete. Hint: Let $\{K_n\}$ be a Cauchy sequence in $\mathcal{K}$ and consider the descending family $H_n = \overline{\bigcup_{j \geq n} K_j}$. Apply Problem 12(c) and show $K_n \to \bigcap_{k \geq 1} H_k$.

    Solution. Google.